SCHWARZIAN DERIVATIVES OF ANALYTIC AND HARMONIC FUNCTIONS

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In memory of Nikolaos Danikas

Abstract. In this expository article, we discuss the Schwarzian derivative of an analytic function and its applications, with emphasis on criteria for univalence and recent results on valence. Generalizations to harmonic mappings are then described, using a definition of Schwarzian recently proposed by the authors. Here it is often natural to identify the harmonic mapping with its canonical lift to a minimal surface.

§1. Historical background.

The *Schwarzian derivative* of a locally univalent analytic function is defined by

$$\mathcal{S}f = (f''/f')' - \tfrac{1}{2}(f''/f')^2 .$$

It has the invariance property $\mathcal{S}(T \circ f) = \mathcal{S}f$ for every Möbius transformation

$$T(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0 .$$

As a special case, $\mathcal{S}(T) = 0$ for every Möbius transformation. If g is any locally univalent analytic function for which the composition $f \circ g$ is defined, then

$$\mathcal{S}(f \circ g) = ((\mathcal{S}f) \circ g)\,(g')^2 + \mathcal{S}g . \tag{1}$$

A function f has Schwarzian $\mathcal{S}f = 2p$ if and only if it has the form $f = u_1/u_2$ for some pair of independent solutions u_1 and u_2 of the linear differential equation $u'' + pu = 0$. As a consequence, if $\mathcal{S}g = \mathcal{S}f$, then $g = T \circ f$ for some Möbius transformation T. In particular, Möbius transformations are the only functions with $\mathcal{S}f = 0$.

The Schwarzian derivative is named for Hermann Amandus Schwarz (1843–1921), but Kummer [13] had used it as early as 1836 in his study of hypergeometric differential equations. Here is the connection. Suppose more generally that we have a differential equation $u'' + qu' + pu = 0$ in the complex plane, where the coefficients

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p and q are rational functions. Let $u_1(z)$ and $u_2(z)$ be independent solutions, defined locally. They are analytic functions, but need not be globally single-valued. As the point z traverses a closed path in the plane, they may return to a different pair of independent solutions, say to $au_1 + bu_2$ and $cu_1 + du_2$. This means that the quotient u_1/u_2 returns to $T(u_1/u_2)$ for some Möbius transformation T. Consequently, the Schwarzian $\mathcal{S}(u_1/u_2)$ is single-valued. In fact, Kummer found the identity

$$\mathcal{S}(u_1/u_2) = 2p - \tfrac{1}{2}\,q^2 - q'\,,$$

which is easily verified.

The Schwarz–Christoffel mapping gives a more or less explicit formula for the conformal mapping of a half-plane onto a polygonal region. Two reflections of the mapping function f produce a function $g = af + b$ for some constants a and b, and the argument hinges on the observation that $g''/g' = f''/f'$. In other words, the *pre-Schwarzian* f''/f' is affine invariant. In 1869, Schwarz [23] generalized the method to construct a conformal mapping of a half-plane onto a region bounded by circular arcs. Here two reflections across circular boundary arcs lead to a function $g = T(f)$ for some Möbius transformation T, and so the Schwarzian derivative is a natural tool because of its Möbius invariance. Nehari's book [16] gives a nice account of this method. The survey article by Osgood [19] contains an extensive historical discussion of Schwarzian derivatives.

In both of the problems just described, it is important to know that the Schwarzian is Möbius invariant. But how could such an operator have been *discovered*? An elegant derivation, attributed to Schwarz, goes as follows. Suppose that

$$g = T(f) = \frac{af+b}{cf+d}\,, \qquad ad - bc \neq 0\,.$$

Then $(cf+d)g = af+b$, and differentiation gives

$$\begin{aligned} c(fg)' + dg' - af' &= 0\,, \\ c(fg)'' + dg'' - af'' &= 0\,, \\ c(fg)''' + dg''' - af''' &= 0\,. \end{aligned}$$

The system of linear homogeneous equations has a nontrivial solution $(c, d, -a)$, so the determinant of coefficients must vanish. After some manipulation, this relation boils down to $\mathcal{S}g = \mathcal{S}f$.

The Schwarzian derivative has a geometric interpretation in terms of curvature. Let C be a smooth curve with parametric representation $z = z(t)$ by arclength t, and let $\kappa = \kappa(t)$ be the curvature of C. Suppose a function $w = f(z)$ maps C conformally to a curve Γ with arclength parametrization $w = w(s)$ and curvature $\widehat{\kappa}(s)$. Then

$$|f'(z(t))|^2\,\frac{d\widehat{\kappa}}{ds} = \frac{d\kappa}{dt} + \mathrm{Im}\left\{(\mathcal{S}f)(z(t))\,z'(t)^2\right\}\,. \qquad (2)$$

Thus the Schwarzian derivative affects the rate of change of curvature with respect to arclength.

In 1949, Zeev Nehari [15] discovered that certain estimates on the Schwarzian imply global univalence. Specifically, he showed that if f is analytic and locally univalent in the unit disk $\mathbb{D}$ and its Schwarzian satisfies either $|\mathcal{S}f(z)| \leq 2(1-|z|^2)^{-2}$ or $|\mathcal{S}f(z)| \leq \pi^2/2$ for all $z \in \mathbb{D}$, then f is univalent in $\mathbb{D}$. Pokornyi [21] then stated, and Nehari proved, that the condition $|\mathcal{S}f(z)| \leq 4(1-|z|^2)^{-1}$ also implies univalence. Nehari [17] unified all three criteria by proving that f is univalent under the general hypothesis $|\mathcal{S}f(z)| \leq 2p(|z|)$, where $p(x)$ is a positive continuous even function defined on the interval $(-1,1)$, with the properties that $(1-x^2)^2 p(x)$ is nonincreasing on the interval $[0,1)$ and no nontrivial solution u of the differential equation $u'' + pu = 0$ has more than one zero in $(-1,1)$. The last condition can be replaced by the equivalent requirement that some solution of the differential equation have no zeros in $(-1,1)$. We will refer to such functions $p(x)$ as *Nehari functions*.

It is clear from the Sturm comparison theorem that if $p(x)$ is a Nehari function, then so is $kp(x)$ for any constant k in the interval $0 < k < 1$. We will call a Nehari function $p(x)$ *extremal* if $kp(x)$ is not a Nehari function for any constant $k > 1$. Chuaqui and Osgood [9] showed that some constant multiple of each Nehari function is an extremal Nehari function. The functions $p(x) = (1-x^2)^{-2}$, $p(x) = \pi^2/4$, and $p(x) = 2(1-x^2)^{-1}$ are all extremal Nehari functions; the nonvanishing solutions are respectively $u = \sqrt{1-x^2}$, $u = \cos(\pi x/2)$, and $u = 1 - x^2$.

The constant 2 in Nehari's general criterion is sharp. To be more precise, suppose $p(x)$ is an extremal Nehari function that is the restriction to the real interval $(-1,1)$ of a function $p(z)$ analytic in $\mathbb{D}$, with the property $|p(z)| \leq p(|z|)$. Then for each constant $C > 2$ the inequality $|\mathcal{S}f(z)| \leq C\,p(|z|)$ admits nonunivalent functions f that are analytic in $\mathbb{D}$. For $p(x) = (1-x^2)^{-2}$ and for any constant $C > 2$, Hille [12] gave an explicit example of an analytic function f that satisfies $|\mathcal{S}f(z)| \leq C(1-|z|^2)^{-2}$, yet has infinite valence in the unit disk. On the other hand, the condition does imply *uniform local univalence* in the sense that any two points where f takes the same value must have a certain minimum separation in the hyperbolic metric. For points $\alpha, \beta \in \mathbb{D}$, the *pseudohyperbolic metric* is

$$\rho(\alpha,\beta) = \left| \frac{\alpha - \beta}{1 - \overline{\alpha}\beta} \right| ,$$

and the *hyperbolic metric* is

$$d(\alpha,\beta) = \tfrac{1}{2} \log \frac{1+\rho(\alpha,\beta)}{1-\rho(\alpha,\beta)} .$$

Binyamin Schwarz [22] proved that if

$$|S(f(z)| \leq \frac{2(1+\delta^2)}{(1-|z|^2)^2} , \qquad z \in \mathbb{D} ,$$

for some constant $\delta > 0$, then $f(\alpha) = f(\beta)$ implies $d(\alpha, \beta) \geq \pi/\delta$. The separation constant π/δ is sharp, as can be shown by Hille's example. Further details are given in our paper [6].

All of these univalence criteria rely ultimately on the Sturm comparison theorem for zeros of solutions to linear differential equations of second order. (The book by Birkhoff and Rota [1] contains a good discussion of the Sturm comparison theorem and related results.) As mentioned earlier, an analytic function f with prescribed Schwarzian $\mathcal{S}f = 2\psi$ has the form $f = u_1/u_2$ for some pair of independent solutions u_1 and u_2 of the equation $u'' + \psi u = 0$. If f takes the same value at two points in the disk, then the differential equation has a solution that vanishes at those two points. Nehari [17] uses a clever trick of conformal mapping to reduce the problem to zeros on the real interval $(-1, 1)$, whereupon the Sturm comparison theorem leads to a contradiction if $\mathcal{S}f$ is suitably restricted.

§2. Valence of analytic functions.

In view of the theorem of Binyamin Schwarz for the Nehari function $p(x) = (1 - x^2)^{-2}$, it is natural to consider the valence of analytic functions f under the more general condition

$$|\mathcal{S}f(z)| \leq C\, p(|z|)\,, \qquad z \in \mathbb{D}\,, \tag{3}$$

where p is an arbitrary (extremal) Nehari function p and $C > 2$. By the *valence* of f we mean $N = \sup_{w \in \mathbb{C}} n(f, w)$, where $n(f, w) \leq \infty$ is the number of points $z \in \mathbb{D}$ for which $f(z) = w$. According to Nehari's theorem, the condition (3) with $C = 2$ implies that f is univalent in the disk. In general the valence will depend on the size of the constant C and the growth of $p(x)$ as $x \to 1$.

At this point it is useful to make some further remarks about Nehari functions. Since $(1 - x^2)^2 p(x)$ is positive and nonincreasing on the interval $(0, 1)$, the limit

$$\mu = \lim_{x \to 1-} (1 - x^2)^2 p(x)$$

exists and $\mu \geq 0$. Observe first that $\mu \leq 1$. Indeed, if $\mu > 1$ then

$$p(x) > \frac{\frac{1}{2}(1 + \mu)}{(1 - x^2)^2}$$

in some interval $x_0 < x < 1$, and the Sturm comparison theorem shows that the solutions of the differential equation $u'' + pu = 0$ have infinitely many zeros in $(-1, 1)$, which is not possible because p is a Nehari function. It is known [6] that the only possibility for $\mu = 1$ is $p(x) = (1 - x^2)^{-2}$.

Thus for $\mu = 1$ the question of valence is covered by the theorems of Nehari and B. Schwarz. For $C = 2$, the condition (3) implies univalence, and for $C > 2$ it admits functions of infinite valence. There is no middle ground. On the other hand, for Nehari functions $p(x)$ other than $(1 - x^2)^{-2}$, so that $\mu < 1$, there is a more gradual transition. In our paper [6] we showed that the condition (3) still implies uniform local univalence when $C > 2$, but if $C\mu < 2$ it implies finite valence. Consequently, if $\mu = 0$, then the condition (3) for *any* constant C implies that f has finite valence. In particular, every analytic function with bounded Schwarzian has finite valence in the unit disk. The same is true more generally if $(1 - |z|^2)|\mathcal{S}f(z)|$ is bounded, a weakened form of Pokornyi's condition.

The last two results can be expressed in quantitative form, as shown in our subsequent paper [7]. There we obtained the following estimates on the valence of an analytic function f that is locally univalent in the disk. If $|\mathcal{S}f(z)| \leq C$ for some constant $C > \pi^2/2$, then $|\alpha - \beta| \geq \sqrt{2/C}\,\pi$ for any pair of points $\alpha, \beta \in \mathbb{D}$ where $f(\alpha) = f(\beta)$. Consequently, f has finite valence and assumes any given value at most $\left(1 + \frac{\sqrt{2C}}{\pi}\right)^2$ times. If $|\mathcal{S}f(z)| \leq C(1 - |z|^2)^{-1}$ for some constant $C > 4$, then f has finite valence $N = N(C) \leq A\,C \log C$, where A is some absolute constant.

§3. Schwarzians of harmonic functions.

It turns out that many of the results for analytic functions can be generalized to harmonic functions, with a suitable definition of Schwarzian derivative. A few years ago, we proposed [2] a notion of Schwarzian derivative for harmonic mappings. The definition was adapted from a very general concept of Schwarzian for mappings between Riemannian manifolds, as developed by Osgood and Stowe [20], by identifying a harmonic mapping with its canonical lift to a minimal surface. Before stating the more general definition, we need to recall some basic facts about harmonic mappings and minimal surfaces.

A planar harmonic mapping is a complex-valued harmonic function

$$f(z) = u(z) + iv(z)\,, \qquad z = x + iy\,,$$

defined on some domain $\Omega \subset \mathbb{C}$. If Ω is simply connected, the mapping has a canonical decomposition $f = h + \overline{g}$, where h and g are analytic in Ω and $g(z_0) = 0$ for some specified point $z_0 \in \Omega$. By a theorem of Hans Lewy, the function f is locally univalent if and only if its Jacobian $J = |h'|^2 - |g'|^2$ does not vanish. It is said to be orientation-preserving if $J(z) > 0$ in Ω, or equivalently if $h'(z) \neq 0$ and the *dilatation* $\omega = g'/h'$ has the property $|\omega(z)| < 1$ in Ω.

According to the Weierstrass–Enneper formulas, a harmonic mapping $f = h + \overline{g}$ with $|h'(z)| + |g'(z)| \neq 0$ lifts locally to a minimal surface described by conformal parameters if and only if its dilatation has the form $\omega = q^2$ for some meromorphic function q. The Cartesian coordinates (U, V, W) of the surface are then given by

$$U(z) = \operatorname{Re}\{f(z)\}\,, \quad V(z) = \operatorname{Im}\{f(z)\}\,, \quad W(z) = 2\operatorname{Im}\left\{\int_{z_0}^{z} \sqrt{h'(\zeta)g'(\zeta)}\,d\zeta\right\}\,.$$

We use the notation

$$\widetilde{f}(z) = \big(U(z), V(z), W(z)\big)$$

for the lifted mapping from Ω to the minimal surface. The first fundamental form of the surface is $ds^2 = \lambda^2|dz|^2$, where the conformal metric is $\lambda = |h'| + |g'|$. The Gauss curvature of the surface at a point $\widetilde{f}(z)$ for which $h'(z) \neq 0$ is

$$K = -\frac{1}{\lambda^2}\Delta(\log\lambda) = -\frac{4|q'|^2}{|h'|^2(1+|q|^2)^4}\,,$$

where Δ is the Laplacian operator. Further information about harmonic mappings and their relation to minimal surfaces can be found in the book [11].

For a harmonic mapping $f = h + \overline{g}$ with $\lambda(z) = |h'(z)| + |g'(z)| \neq 0$, whose dilatation is the square of a meromorphic function, the *Schwarzian derivative* is defined [2] by the formula

$$\mathcal{S}f = 2\big(\sigma_{zz} - {\sigma_z}^2\big)\,, \qquad \sigma = \log\lambda\,,$$

where

$$\sigma_z = \frac{\partial\sigma}{\partial z} = \frac{1}{2}\left(\frac{\partial\sigma}{\partial x} - i\frac{\partial\sigma}{\partial y}\right)\,, \qquad z = x + iy\,.$$

If f is analytic, it is easily verified that $\mathcal{S}f$ reduces to the classical Schwarzian. If f is harmonic and φ is analytic, then $f\circ\varphi$ is harmonic and

$$\mathcal{S}(f\circ\varphi) = ((\mathcal{S}f)\circ\varphi)(\varphi')^2 + \mathcal{S}\varphi\,,$$

a generalization of the classical composition formula (1). With $h'(z) \neq 0$ and $g'/h' = q^2$, a calculation (*cf.* [2]) produces the expression

$$\mathcal{S}f = \mathcal{S}h + \frac{2\overline{q}}{1+|q|^2}\left(q'' - q'\,\frac{h''}{h'}\right) - 4\left(\frac{q'\overline{q}}{1+|q|^2}\right)^2\,.$$

It was shown in [2] that a locally sense-preserving harmonic mapping f has an analytic Schwarzian $\mathcal{S}f$ if and only if it has the form $f = h + \alpha\overline{h}$ for some analytic locally univalent function h and some constant α with $|\alpha| < 1$. It is easy to see that $\mathcal{S}(h+\alpha\overline{h}) = \mathcal{S}h$, which shows that $\mathcal{S}(T+\alpha\overline{T}) = 0$ for every Möbius transformation T. In fact, the converse is also true, so that f has Schwarzian $\mathcal{S}f \equiv 0$ if and only if $f = T + \alpha\overline{T}$ for $|\alpha| < 1$ and some Möbius transformation T. Such functions $f = T + \alpha\overline{T}$ are called *harmonic Möbius transformations.* They take circles to ellipses. Conversely, it has been shown [3] that harmonic Möbius transformations are the only sense-preserving harmonic mappings that carry circles to ellipses.

The curvature property (2) of Schwarzian derivatives also generalizes to harmonic mappings, but now we must consider the Weierstrass–Enneper lift $\widetilde{f}$ of the harmonic mapping to a minimal surface and consider the *geodesic* curvature of the image curve with respect to the minimal surface. This relation is contained in the general formulation by Osgood and Stowe [20], but a more elementary derivation was given in [4] without appeal to concepts of Riemannian geometry.

It must be emphasized that the requirement $|h'(z)| + |g'(z)| \neq 0$ does not entail the local univalence of our harmonic mappings $f = h + \overline{g}$. In other words, the Jacobian need not be of constant sign in the domain Ω. The orientation of the mapping may reverse, corresponding to a folding in the associated minimal surface. It is also possible for the minimal surface to exhibit several sheets above a point in the (U, V)–plane. Thus the lifted mapping $\widetilde{f}$ may be locally or globally univalent even when the underlying mapping f is not.

Elsewhere [5] we have developed the following criterion for the lift of a harmonic mapping to be univalent.

Theorem. *Let $f = h + \overline{g}$ be a harmonic mapping of the unit disk, with $\lambda(z) = |h'(z)| + |g'(z)| \neq 0$ and dilatation $g'/h' = q^2$ for some meromorphic function q. Let $\widetilde{f}$ denote the Weierstrass–Enneper lift of f to a minimal surface with Gauss curvature $K = K(\widetilde{f}(z))$ at the point $\widetilde{f}(z)$. Suppose that the inequality*

$$|\mathcal{S}f(z)| + \lambda(z)^2 |K(\widetilde{f}(z))| \leq 2p(|z|)\,, \qquad z \in \mathbb{D}\,, \tag{4}$$

holds for some Nehari function p. Then $\widetilde{f}$ is univalent in $\mathbb{D}$.

If f is analytic, its associated minimal surface is the complex plane itself, with Gauss curvature $K = 0$, and the result reduces to Nehari's theorem. Under the hypotheses of the theorem, it turns out that the Weierstrass–Enneper lift $\widetilde{f}$ has an extension to $\overline{\mathbb{D}}$ that is continuous with respect to the spherical metric. In any extremal case, where the image $\widetilde{f}(\partial\mathbb{D})$ of the boundary is not a simple curve on the minimal surface, it can be shown that the minimal surface is part of a catenoid or a plane, and equality holds in (3) along the preimage of a circle of revolution of the catenoid or a line on the plane. Explicit examples were given in [5] to show that the result is sharp for the Nehari functions $p(x) = \pi^2/4$, $(1 - x^2)^{-2}$, and $2(1 - x^2)^{-1}$.

The proof of the theorem is based on a recent result of Chuaqui and Gevirtz [8], which refers to the *Ahlfors Schwarzian*

$$S_1\varphi = \frac{\langle \varphi', \varphi''' \rangle}{|\varphi'|^2} - 3\,\frac{\langle \varphi', \varphi'' \rangle^2}{|\varphi'|^4} + \tfrac{3}{2}\,\frac{|\varphi''|^2}{|\varphi'|^2}$$

of a curve $\varphi : (-1, 1) \to \mathbb{R}^n$. This definition arises by imitating the expression for the real part of the Schwarzian of an analytic function. Chuaqui and Gevirtz proved that if $\varphi : (-1, 1) \to \mathbb{R}^n$ is a curve of class C^3 with $\varphi'(x) \neq 0$, and if $S_1\varphi(x) \leq 2p(x)$ for some Nehari function p, then φ is univalent on $(-1, 1)$. For the restriction $\widetilde{f} : (-1, 1) \to \mathbb{R}^3$ of the harmonic lift to the real interval, it is shown in [5] that

$$S_1\widetilde{f}(x) \leq \mathrm{Re}\{\mathcal{S}f(x)\} + \lambda(x)^2 |K(\widetilde{f}(x))|\,.$$

Thus the hypothesis (4), combined with the result of Chuaqui and Gevirtz, implies that $\widetilde{f}$ is univalent on the interval $(-1, 1)$. The "Nehari trick" (*cf.* [17]) is then adapted to deduce univalence in the whole disk. Actually, the result of Chuaqui and Gevirtz does not require $p(x)$ to be a Nehari function; it is enough that the differential equation $u'' + pu = 0$ have no nontrivial solution with more than one zero on $(-1, 1)$. The nonincreasing property of $(1 - x^2)^2 p(x)$ is needed only for the Nehari trick.

If the bound (4) is relaxed to

$$|\mathcal{S}f(z)| + \lambda(z)^2 |K(\widetilde{f}(z))| \leq C\, p(|z|)\,, \qquad z \in \mathbb{D}\,,$$

for some constant $C > 2$, it is possible to show [6] as in the analytic case that $\widetilde{f}$ is uniformly locally univalent, and that $\widetilde{f}$ has finite valence if $C\mu < 2$. Quantitative estimates on the valence of $\widetilde{f}$ can also be obtained [7] in terms of the constant C and the Nehari function p, by essentially the same methods that were applied to analytic functions.

Finally, it is natural to inquire about necessary conditions for univalence. For a harmonic mapping f whose dilatation is the square of a meromorphic function, we can define the *Schwarzian norm*

$$\|\mathcal{S}f\| = \sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |\mathcal{S}f(z)|\,.$$

Because the composition formula (1) generalizes to harmonic mappings, it is not difficult to see that the Schwarzian norm retains the Möbius invariance property $\|\mathcal{S}(f \circ \varphi)\| = \|\mathcal{S}f\|$ it enjoys for analytic functions. Here φ is any Möbius self-mapping of the disk. If f is an analytic locally univalent function in the disk, the first theorem of Nehari [15] says that the inequality $\|\mathcal{S}f\| \leq 2$ implies global univalence. Conversely, it is a result of Kraus [14] (rediscovered by Nehari) that if an analytic function f is univalent in $\mathbb{D}$, then $\|\mathcal{S}f\| \leq 6$. The Koebe function $k(z) = z(1 - z)^{-2}$, with Schwarzian $\mathcal{S}k(z) = -6(1 - z^2)^{-2}$, shows that the bound 6 is best possible. We recently proved [7] that the Schwarzian norms of univalent harmonic mappings are finite and have the uniform bound $\|\mathcal{S}f\| < 19,407$. It is an open problem to find the best bound. The "harmonic shear" of the Koebe function with dilatation z^2 has Schwarzian norm 16, so it is reasonable to conjecture that $\|\mathcal{S}f\| \leq 16$ for all univalent harmonic mappings f whose dilatation is a perfect square.

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